Econ 802
First Midterm
Answer Key

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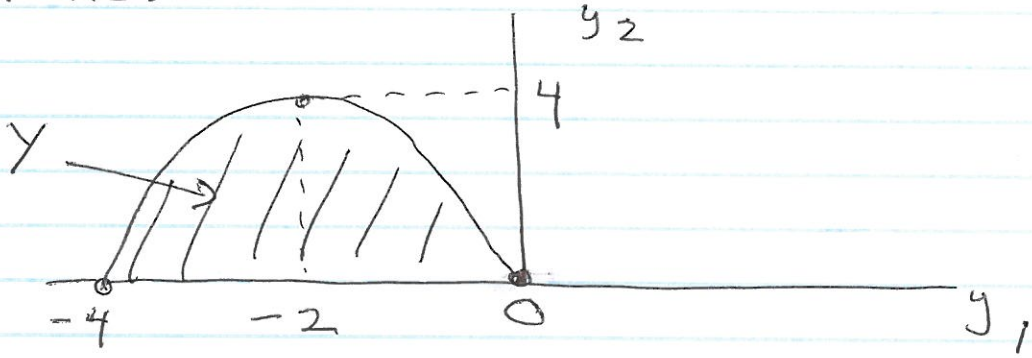
1. (a) The maximum output from y_1 is $4-(y_1+2)^2$. This is zero at $y_1=0$ and $y_1=-4$. It reaches a maximum of 4 at $y_1=-2$. Clearly it is quadratic:



Y is closed because it contains its boundary.
It is bounded because there is a circle of finite diameter centered on the origin that contains all points in Y.
It is not strictly convex because the lower boundary is a straight line [a convex combination of (0,0) and (−4,0) gives a point on the boundary not in the interior of Y].

(b) $V(0) = \{\text{all } x \geq 0 \text{ such that } 4-(2-x)^2 \geq 0\} = [0, 4]$

$V(2) = \{\text{all } x \geq 0 \text{ such that } 4-(2-x)^2 \geq 2\} = [2-\sqrt{2}, 2+\sqrt{2}]$

$V(4) = \{\text{all } x \geq 0 \text{ such that } 4-(2-x)^2 \geq 4\} =$ the single point $x = 2$.

None of these sets are monotone because $x \in V(y)$ does not imply $x' \in V(y)$ for all $x' \geq x$ (due to upper bounds)

(c) Profit max always has a solution. The set Y is non-empty and compact (closed and bounded). The function to be maximized is continuous. By the Weierstrasse Theorem there is some $y^* \in Y$ that maximizes profit at the given prices.

Cost min sometimes has a solution. If $0 \le y \le 4$ there is a solution because $V(y)$ is non-empty and we can choose the smallest x that gives y. But if $y > 4$ there is no solution because $V(y)$ is empty.

2(a) A necessary condition for profit max is cost min. Cost min $\Rightarrow y = ax_1 = bx_2 \Rightarrow x_1 = \frac{y}{a}$ and $x_2 = \frac{y}{b}$. So $c(w,y) = y\left(\frac{w_1}{a} + \frac{w_2}{b}\right)$. Note that AC is a constant due to CRS: $AC = \frac{w_1}{a} + \frac{w_2}{b}$. There is no solution if $p > \frac{w_1}{a} + \frac{w_2}{b}$ because profit can be made arbitrarily large by increasing y. If $p = \frac{w_1}{a} + \frac{w_2}{b}$ there is a solution but it is non-unique because any $y \ge 0$ gives zero profit. If $p < \frac{w_1}{a} + \frac{w_2}{b}$ then the unique solution has $y = 0$, $x_1 = 0$, $x_2 = 0$, which gives zero profit.

(b) For a given y, cost min $\Rightarrow x_1 = 0,\ x_2 = \frac{y}{b}$ if $\frac{w_1}{w_2} > \frac{a}{b}$

$x_1 = \frac{y}{a},\ x_2 = 0$ if $\frac{w_1}{w_2} < \frac{a}{b}$

and any (x_1, x_2) on the isoquant if $\frac{w_1}{w_2} = \frac{a}{b}$.

(1) So when $\frac{w_1}{w_2} \ge \frac{a}{b}$ we have $c(w,y) = \frac{w_2}{b} y$;

(2) When $\frac{w_1}{w_2} \le \frac{a}{b}$ we have $c(w,y) = \frac{w_1}{a} y$

Again CRS $\Rightarrow$ AC is constant. There is no solution if $p > \frac{w_2}{b}$ when $\frac{w_1}{w_2} \ge \frac{a}{b}$ or $p > \frac{w_1}{a}$ when $\frac{w_1}{w_2} \le \frac{a}{b}$.

There is a solution when case 1 holds and $p = \frac{w_2}{b}$ or when case 2 holds and $p = \frac{w_1}{a}$, but it is not unique; any $y \geq 0$ gives zero profit. There is a unique solution in case 1 if $p < \frac{w_1}{a}$ or in case 2 when $p < \frac{w_2}{b}$. In each case the solution is $y = 0$ which gives zero profit.

(c) Profit max definitely has a solution when $a+b<1$ so we have DRS. A fast argument goes as follows. We said in class that the cost function has the form

$$c(w,y) = c(w,1)\, y^{\frac{1}{a+b}}$$

So the firm wants to maximize $py - c(w,y)$

or $py - c(w,1)\, y^{\frac{1}{a+b}}$

Differentiate with respect to output to get

FOC: $p - c(w,1)\, y^{\frac{1}{a+b}-1} = 0.$

The solution for y is unique and positive. It is easy to check the sufficient SOC which is

SOC: $-c(w,1)\left[\frac{1}{a+b} - 1\right] y^{\frac{1}{a+b}-2} < 0$

This holds when $a+b<1$ so the solution to FOC is indeed a maximizer.

Profit max definitely does not have a solution when $a+b>1$, which implies IRS. You can use a similar argument to the one above. The FOC has a unique solution, so there is only one candidate for a maximizer. However the necessary SOC is violated because the second derivative is positive when it needs to be non positive. This shows that there is no solution.

[Note: when $a+b=1$ so we have CRS, the existence of a solution depends on the details about the prices.]

3.(a) We need to show $\pi[tp + (1-t)p'] \le t\pi(p) + (1-t)\pi(p')$ for $0 \le t \le 1$ (This is the definition of a convex function). Define $p'' \equiv tp + (1-t)p'$ and let y'' be the optimal production plan for p''. Then

$$\pi[tp + (1-t)p'] = p''y'' = tpy'' + (1-t)p'y''.$$

y'' may or may not be optimal at prices p, and likewise y'' may or may not be optimal at prices p'. In any case, $py'' \le \pi(p)$ and $p'y'' \le \pi(p')$. This shows that $\pi[tp + (1-t)p'] \le t\pi(p) + (1-t)\pi(p')$.

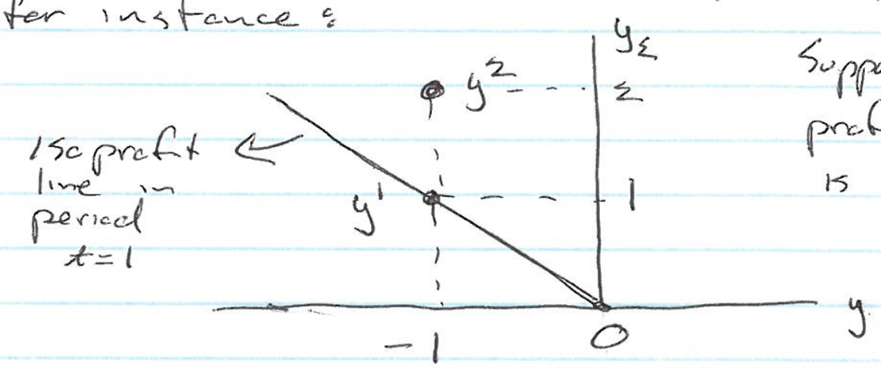
(b) Hotelling's Lemma says that $\frac{\partial \pi(p)}{\partial p_i} = y_i(p)$, $i = 1 \ldots n$. (assuming $p > 0$ and the derivatives exist)

Proof: Let y^* be optimal at prices p^*. Define
$g(p) = \pi(p) - py^* \ge 0$ where non-negativity follows from the fact that maximum profit at p must be at least py^* (note that y^* may or may not be optimal for some arbitrary p). Also $g(p^*) = 0$ because we know y^* is optimal at p^* so $\pi(p^*) = p^*y^*$. Therefore $g(p)$ reaches a minimum at p^* and the FOC for a minimum must hold at this point $\Rightarrow$ $\frac{\partial g(p^*)}{\partial p_i} = \frac{\partial \pi(p^*)}{\partial p_i} - y_i^* = 0$

However, p^* was arbitrary, so in general

$$\frac{\partial \pi(p)}{\partial p_i} = y_i(p) \quad \text{where } y_i(p) \text{ is optimal for } p.$$
$(i = 1 \ldots n)$

(c) Suppose you observe (p^1, y^1) at $t=1$. If you see <u>any</u> y^2 above the iso-profit line in period 1 this contradicts WAPM, because the firm could have chosen y^2 in period $t=1$ and this would have given higher profit than p^1y^1. Notice that it doesn't matter here what the prices p^2 are, so for instance:



Suppose $p_1^1 = p_2^1 = 1$ and profit in period 1 is zero due to $y^1 = (-1, +1)$.

If you observe y^2 in period $t=2$ this contradicts WAPM because the firm was not maximizing profit when it chose y^1.

4 (a) If we assume (ii) instead of (i) we know that any point satisfying FOC actually <u>does</u> minimize cost [if all we have is (i), we can't be sure of this].

If we assume (iii) instead of (ii), we know the solution of the cost-min problem is <u>unique</u> [we don't know this if all we have is (ii)].

If we assume (iv) instead of (iii) we know that we can use the implicit function theorem to differentiate FOC and get the derivatives of the conditional factor demands with respect to prices [(iii) alone does not guarantee that we can do this] (b). This issue would arise if the economist is trying to differentiate the FOC for cost minimization in order to find out how the conditional factor demands $x(w,y)$ vary with the prices w.

FOC: $w = \lambda \dfrac{\partial f(x)}{\partial x}$. Let the solution be $x(w)$ and $\lambda(w)$

$f(x) = y$ (ignore variations in y)

Treat FOC as an identity: $w \equiv \lambda(w) \dfrac{\partial f[x(w)]}{\partial x}$

$f[x(w)] \equiv y$

Now differentiate with respect to the vector w:

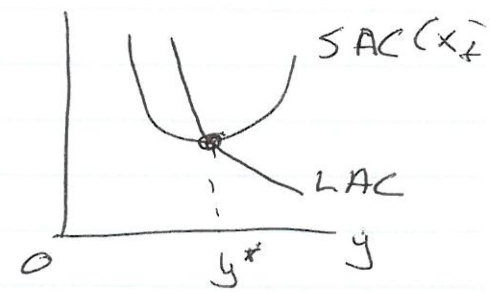
$$\lambda(w) \frac{\partial^2 f[x(w)]}{\partial x^2} \frac{\partial x(w)}{\partial w} + \frac{\partial f[x(w)]}{\partial x} \frac{\partial \lambda(w)}{\partial w} = I$$

$$\frac{\partial f[x(w)]}{\partial x} \frac{\partial x(w)}{\partial w} = \underline{0}$$

$$\Rightarrow \begin{bmatrix} \lambda(w) \dfrac{\partial^2 f[x(w)]}{\partial x^2} & \dfrac{\partial f[x(w)]}{\partial x} \\ \dfrac{\partial f[x(w)]}{\partial x} & 0 \end{bmatrix} \begin{bmatrix} \dfrac{\partial x(w)}{\partial w} \\ \dfrac{\partial \lambda(w)}{\partial w} \end{bmatrix} = \begin{bmatrix} I \\ \underline{0} \end{bmatrix}$$

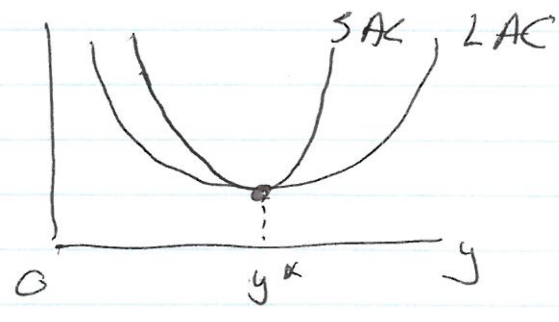
At this point we would like to invert the bordered Hessian on the left hand side in order to solve for $\frac{\partial x}{\partial w}$, the substitution matrix. The usual conclusions are that $\frac{\partial x}{\partial w}$ is symmetric and negative definite.

(c) Suppose LAC is decreasing as it passes through the minimum point of SAC:



This makes no sense because to the left of y^* we have
$LAC(y) > SAC(y)$ or
$$\frac{c(w,y)}{y} > \frac{c(w,y,x_f)}{y}$$
$$\Rightarrow c(w,y) > c(w,y,x_f).$$
But this says that cost is <u>higher</u> in the long run than in the short run, which is false. In the long run all inputs are variable, so it would always be possible to replicate the short run plan. The same argument applies if LAC is <u>rising</u> as it passes through the minimum point of SAC.

One exception: suppose LAC is U-shaped, and the minimum point of SAC occurs at the same output as the minimum point of LAC. Then we get this:



No contradiction in this case, because $SAC \geq LAC$ at all output levels.

5(a) Definition: $$\sigma = -\frac{\partial\left(\frac{x_1}{x_2}\right)}{\partial\left(\frac{w_1}{w_2}\right)} \cdot \frac{\left(\frac{w_1}{w_2}\right)}{\left(\frac{x_1}{x_2}\right)}$$

I would tell the student that this concept is useful in thinking about how the expenditures of the firm are distributed among the suppliers of different inputs (like capital and labor), and how the distribution changes as input prices change.

For instance, suppose x_1 = labor, x_2 = capital, and there is a change in the price ratio w_1/w_2.

If $\frac{w_1 x_1}{w_2 x_2} = s$ goes up, labor gets a larger share in total expenditure.

Define $\hat{w} = \frac{w_1}{w_2}$, $\hat{x} = \frac{x_1}{x_2}$, so $s = \hat{w}\hat{x}$.

Then $$\frac{\partial s}{\partial \hat{w}} = \hat{x} + \hat{w}\frac{\partial \hat{x}}{\partial \hat{w}} = \hat{x} \cdot \left[1 + \frac{\partial \hat{x}}{\partial \hat{w}} \cdot \frac{\hat{w}}{\hat{x}}\right] = \hat{x}(1-\sigma)$$

So if $\frac{w_1}{w_2} \uparrow$ then labor's share goes up when $\sigma < 1$ and capital's share goes up when $\sigma > 1$.

The key issue is whether the firm is very sensitive to price changes (elastic case, $\sigma > 1$) or not very sensitive (inelastic case, $\sigma < 1$).

I would probably also tell the student that all of this is based on the assumptions that ① the firm is a price-taker in its input markets; ② the firm is minimizing cost; and ③ output is held constant.

(b) A homogeneous function has $f(tx) = t^k f(x)$ for some $k \geq 0$. Differentiate with respect to t:

$$\sum_{i=1}^{n} \frac{\partial f(tx)}{\partial x_i} \cdot x_i = k t^{k-1} f(x).$$ Then set $t = 1$ to get

$$\sum_{i=1}^{n} \frac{\partial f(x)}{\partial x_i} \cdot x_i = k f(x) \Rightarrow e(x) = k$$ where $e(x)$ is

the local elasticity with respect to scale at x.

Because $e(x) = k$ is a constant, we either have ① $k > 1 \Rightarrow$ IRS everywhere; ② $k = 1 \Rightarrow$ CRS everywhere; or ③ $k < 1 \Rightarrow$ DRS everywhere. In case ①, LAC is always falling; in ② it is horizontal; and in ③ it is always rising. So none of these gives U-shaped LAC.

(c) You could use the algebraic approach based on WAPM or you could differentiate the FOC. I think the easiest method is based on Hotelling's Lemma. Define

$$\pi(p, w) \equiv \max_{x \geq 0} \{ p f(x) - wx \}$$

From Hotelling, we have $\dfrac{\partial \pi(p,w)}{\partial p} = y(p,w)$.

We also know that $\pi(p,w)$ is convex. So holding w constant, it must be true that

$$\frac{\partial^2 \pi(p,w)}{\partial p^2} = \frac{\partial y(p,w)}{\partial p} \geq 0.$$

Therefore output $y(p,w)$ cannot decrease when the output price increases, holding input prices constant.

[The output supply curve of the firm cannot slope down.]